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From the Classical Field with Sources to the Multipole Matrix Elements

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Starting with the classical interaction of the electromagnetic field with sources and then quantizing, the interaction Hamiltonian is expanded in multipoles. Explicit expressions for the multipole operators are obtained. Transformation properties, under time reversal and Hermitian conjugation, of nuclear states and multipole operators are studied.

1. INTRODUCTION

An important part of the information on nuclear properties has been extracted from studies of electromagnetic transitions.

In this article we start with the well-understood classical interaction of the electromagnetic field with charges and currents. The classical equations are quantized and the interaction Hamiltonian expanded in terms of multipoles. The explicit expressions of the electromagnetic multipole operators will be obtained and properties of their matrix elements studied. One-photon states of definite angular momentum will be described since they play an important role in nuclear phenomena. The formalism is derived from first principles, and we have tried to make the discussion reasonably self-contained. Special reference will be made to phase conventions, and the transformation properties of nuclear states and multipole operators, under time reversal and Hermitian conjugation, will be discussed.

The contents of the paper are as follows:

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4. Quantization of the Free Radiation Field
5. The Interaction Hamiltonian between the Radiation Field and Nonrelativistic Charges
6. The Nucleus as a System of Nonrelativistic Spin-$\frac{1}{2}$ Particles
7. Emission of Electromagnetic Radiation
8. The Angular Momentum and Parity of the Photon
9. The Multipole Operators
10. The Behavior of the Multipole Operators under Hermitian Conjugation and Time Reversal: Reality of the Matrix Elements

2. THE DESCRIPTION OF THE FREE RADIATION FIELD IN HAMILTONIAN FORM

A useful gauge to study radiation phenomena is the Coulomb gauge

$$\operatorname{div} \mathbf{A} = 0 \tag{2.1}$$

The so-called radiation or transverse fields are derived from a vector potential satisfying this relation. Furthermore, in a charge-free region, it is possible to transform the scalar potential ϕ to zero. The electromagnetic field can then be described by the vector potential $\mathbf{A}(\mathbf{r}, t)$, satisfying the wave equation

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = 0 \tag{2.2}$$

The electric and magnetic fields are then given by

$$\mathscr{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \tag{2.3}$$

$$\mathscr{H} = \operatorname{curl} \mathbf{A} \tag{2.4}$$

This is essentially a description using a continuous set of variables. Keeping in mind that our purpose is the quantization of the electromagnetic field (Section 4) a more convenient way of describing the electromagnetic field, in Hamiltonian form, is by confining the field in a large cube of volume $V = L^3$. Requiring periodic boundary conditions on opposite faces of the cube, we can expand $\mathbf{A}(\mathbf{r}, t)$ in terms of a set of discrete oscillating modes with different characteristic frequencies. With $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ the unit vectors along the edges, the required boundary conditions

$$\mathbf{A}(\mathbf{r} + L\hat{\mathbf{i}}, t) = \mathbf{A}(\mathbf{r} + L\hat{\mathbf{j}}, t) = \mathbf{A}(\mathbf{r} + L\hat{\mathbf{k}}, t) = \mathbf{A}(\mathbf{r}, t)$$

are satisfied if the wave vector

$$\mathbf{k} = \frac{2\pi}{L} (n_x \hat{\mathbf{i}} + n_y \hat{\mathbf{j}} + n_z \hat{\mathbf{k}})$$

is restricted in such a way that each n may only take integer values from $-\infty$ to $+\infty$.

Expanding, at a given instant, in Fourier series in that cube

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\mathbf{k}} \sum_{\sigma=1,2} \left(\frac{2\pi\hbar c^2}{\omega_k L^3}\right)^{1/2} [a_{\mathbf{k}\sigma}(t)\hat{\mathbf{e}}_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r}} + a^*_{\mathbf{k}\sigma}(t)\hat{\mathbf{e}}_{\mathbf{k}\sigma} e^{-i\mathbf{k}\cdot\mathbf{r}}] \quad (2.5)$$

stressing that $\mathbf{A}$ is real. The Fourier coefficients

$$a_{\mathbf{k}\sigma}(t) = a_{\mathbf{k}\sigma}(0) e^{-i\omega_k t} \quad (2.6)$$

with $\omega_k = |\mathbf{k}|c$, are a discrete set of variables characterizing the field and $(2\pi\hbar c^2/\omega_k L^3)^{1/2}$ a convenient normalization coefficient.

The expansion in terms of the complete orthonormal set of plane waves $L^{-3/2}\hat{\mathbf{e}}_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r}}$ is just a convenient mathematical device (depending on the geometry of the box and boundary conditions), and does not involve a restriction to the problem. The unit vectors $\hat{\mathbf{e}}_{\mathbf{k}\sigma}$ are the polarization vectors, and the Coulomb gauge condition (2.1) demands that $\mathbf{A}(\mathbf{r}, t)$ is a transverse vector.

From (2.3) one may easily find

$$\mathscr{E}(\mathbf{r}, t) = \frac{i}{c} \sum_{\mathbf{k}} \sum_{\sigma=1,2} \left(\frac{2\pi\hbar c^2}{\omega_k L^3}\right)^{1/2} \omega_k [a_{\mathbf{k}\sigma}(t)\hat{\mathbf{e}}_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r}} - a^*_{\mathbf{k}\sigma}(t)\hat{\mathbf{e}}_{\mathbf{k}\sigma} e^{-i\mathbf{k}\cdot\mathbf{r}}] \quad (2.7)$$

and from, (2.4)

$$\mathscr{H}(\mathbf{r}, t) = i \sum_{\mathbf{k}} \sum_{\sigma=1,2} \left(\frac{2\pi\hbar c^2}{\omega_k L^3}\right)^{1/2} \mathbf{k} \times [a_{\mathbf{k}\sigma}(t)\hat{\mathbf{e}}_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r}} - a^*_{\mathbf{k}\sigma}(t)\hat{\mathbf{e}}_{\mathbf{k}\sigma} e^{-i\mathbf{k}\cdot\mathbf{r}}] \quad (2.8)$$

Using (2.7) and (2.8) we find for the energy of the field

$$H = \frac{1}{8\pi} \int_{\text{cube}} (\mathscr{E}^2 + \mathscr{H}^2)\, d^3\mathbf{r} = \frac{1}{2} \sum_{\mathbf{k}} \sum_{\sigma=1,2} \hbar\omega_k (a_{\mathbf{k}\sigma} a^*_{\mathbf{k}\sigma} + a^*_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma}) \quad (2.9)$$

showing that H is a conserved quantity. Although classically $a_{\mathbf{k}\sigma}$ and $a^*_{\mathbf{k}\sigma}$ commute, we kept the order aa^* and a^*a because later we shall extend the interpretation to noncommuting operators.

To show clearly the analogy with a set of uncoupled oscillators, thus justifying the oscillator expansion of the field, it is convenient to replace the noncanonical variables $a_{\mathbf{k}\sigma}$ and $a^*_{\mathbf{k}\sigma}$ by the real linear combinations

$$Q_{\mathbf{k}\sigma}(t) = \left(\frac{\hbar}{2\omega_k}\right)^{1/2} [a^*_{\mathbf{k}\sigma}(t) + a_{\mathbf{k}\sigma}(t)] \quad (2.10)$$

$$P_{\mathbf{k}\sigma}(t) = i\omega_k \left(\frac{\hbar}{2\omega_k}\right)^{1/2} [a^*_{\mathbf{k}\sigma}(t) - a_{\mathbf{k}\sigma}(t)] \quad (2.11)$$

We obtain from (2.9)

$$H = \tfrac{1}{2} \sum_{\mathbf{k}} \sum_{\sigma=1,2} (P_{\mathbf{k}\sigma}^2 + \omega_k{}^2 Q_{\mathbf{k}\sigma}^2) \tag{2.12}$$

Calculating

$$\frac{\partial H}{\partial Q_{\mathbf{k}\sigma}} = \omega_k{}^2 Q_{\mathbf{k}\sigma} = -\dot{P}_{\mathbf{k}\sigma}$$

and (2.13)

$$\frac{\partial H}{\partial P_{\mathbf{k}\sigma}} = P_{\mathbf{k}\sigma} = \dot{Q}_{\mathbf{k}\sigma}$$

we see that H corresponds to the Hamiltonian for the field and that $P_{\mathbf{k}\sigma}$ and $Q_{\mathbf{k}\sigma}$ are canonically conjugate momenta and coordinates.

Using the index λ for each mode associated with a wave vector $\mathbf{k}$ and a polarization σ

$$H = \sum_{\lambda} H_\lambda \tag{2.14}$$

where H_λ reminds us of the Hamiltonian of the oscillator in classical mechanics, of frequency ω_k and unit mass.

Thus, the radiation field behaves, formally, as an infinite set of independent radiation oscillators. In Section 4 this description of the radiation field in Hamiltonian form will make simple the quantization of the field, by analogy with the classical problem, replacing the dynamical variables P and Q by operators and imposing on them the usual commutation rules.

3. THE DESCRIPTION OF THE FIELD WITH CHARGES, IN HAMILTONIAN FORM

The nonrelativistic Hamiltonian for a system of particles of charge e_k, described by the canonical variables q_k and p_k, in a field having the potentials $\phi(\mathbf{r}_k, t)$ and $\mathbf{A}(\mathbf{r}_k, t)$ at the position of the kth particle, has the form

$$H = \sum_k \left\{ \frac{1}{2m_k} \left[\mathbf{p}_k - \frac{e_k}{c} \mathbf{A}(\mathbf{r}_k, t) \right]^2 + e_k \phi(\mathbf{r}_k, t) \right\} \tag{3.1}$$

It should be noted that in ϕ and $\mathbf{A}$ contributions of two sources are included. One is made up of all the charges of our system (which we shall call system A). The other consists of external sources (which we shall call system B), whose motion we shall assume to be known. We shall suppose the whole system (A + B) to be closed. What we loosely call H_A, and is in fact the Hamiltonian for system A in the presence of system B, may be found from the Hamiltonian for the two coupled systems if we take into account

that the motion of system B corresponds to known functions of time (Landau and Lifshitz, 1960). So, in some aspects of the use of the Hamiltonian (3.1) we may "eliminate" the contribution brought in by the external field. This does not mean that the motion of system B is supposed to be independent of the motion of system A, but only that the motion of system B is assumed to be given by known functions of time. As we are, for the moment, interested in applications involving Hamilton's equations, we may, temporarily, dispense with the contribution of the external potential (ϕ^e, $\mathbf{A}^e$) to the Hamiltonian (3.1). In this way we shall obtain first results concerning only the field produced by the particles of our system and later, in the final result, we can reintroduce the external potential.

The potentials ϕ and $\mathbf{A}$ can be made to satisfy the Lorentz gauge (which is in some way a kind of canonical transformation in the sense that it leaves Maxwell's equations invariant)

$$\operatorname{div} \mathbf{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0 \tag{3.2}$$

Using this gauge we obtain the following uncoupled equations equivalent to Maxwell's equations:

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -4\pi\rho \tag{3.3}$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{j} \tag{3.4}$$

Next we shall write these equations in the usual canonical form of classical dynamics, since we are aiming at the transition to quantum mechanics. It is known that a vector field $\mathbf{A}$ can be split in longitudinal and transverse parts

$$\mathbf{A} = \mathbf{A}^T + \mathbf{A}^L \tag{3.5}$$

such that

$$\operatorname{div} \mathbf{A}^T = 0 \tag{3.6}$$

$$\operatorname{curl} \mathbf{A}^L = 0 \tag{3.7}$$

Assuming the field confined in a large cube we may develop into Fourier series (detailed discussion in Heitler, 1954)

$$\mathbf{A}^T = \sum_{\lambda} Q_\lambda(t)\mathbf{A}_\lambda(\mathbf{r}) \tag{3.8}$$

$$\mathbf{A}^L = \sum_{\sigma} Q_\sigma(t)\mathbf{A}_\sigma(\mathbf{r}) \tag{3.9}$$

$$\phi = \sum_{\sigma} Q_{0\sigma}(t)\phi_\sigma(\mathbf{r}) \tag{3.10}$$

with $\mathbf{A}_\lambda$, $\mathbf{A}_\sigma$, and ϕ_σ representing complete orthogonal sets (respectively, transverse, longitudinal, and scalar functions) satisfying the wave equation and periodic boundary conditions.

Expressing the continuous charge densities in terms of "point" charges e_k at $\mathbf{r}_k$

$$\rho(\mathbf{r}) = \sum_k e_k \delta(\mathbf{r} - \mathbf{r}_k)$$

$$\mathbf{j}(\mathbf{r}) = \sum_k e_k \mathbf{v}_k \delta(\mathbf{r} - \mathbf{r}_k)$$

we may write

$$\int_{\text{cube}} \mathbf{j} \cdot \mathbf{A}_\lambda(\mathbf{r})\, d^3\mathbf{r} = \sum_k e_k \mathbf{v}_k \cdot \mathbf{A}_\lambda(\mathbf{r}_k) \tag{3.11}$$

and

$$\int_{\text{cube}} \rho \phi_\sigma(\mathbf{r})\, d^3\mathbf{r} = \sum_k e_k \phi_\sigma(\mathbf{r}_k) \tag{3.12}$$

and together with the orthogonality conditions for the $\mathbf{A}_\lambda$, $\mathbf{A}_\sigma$, and ϕ_σ we may write for (3.3) and (3.4)

$$\ddot{Q}_\lambda + \omega_\lambda^2 Q_\lambda = \frac{1}{c} \sum_k e_k \mathbf{v}_k \cdot \mathbf{A}_\lambda(\mathbf{r}_k) \tag{3.13}$$

$$\ddot{Q}_\sigma + \omega_\sigma^2 Q_\sigma = \frac{1}{c} \sum_k e_k \mathbf{v}_k \cdot \mathbf{A}_\sigma(\mathbf{r}_k) \tag{3.14}$$

$$\ddot{Q}_{0\sigma} + \omega_\sigma^2 Q_{0\sigma} = \sum_k e_k \phi_\sigma(\mathbf{r}_k) \tag{3.15}$$

We should bring to attention, at this stage, that in Section 2 we saw that the radiation field could be described by canonical variables $Q_{\mathbf{k}\sigma}$ and $P_{\mathbf{k}\sigma}$ and as a consequence of (2.13)

$$\ddot{Q}_{\mathbf{k}\sigma} + \omega_k^2 Q_{\mathbf{k}\sigma} = 0 \tag{3.16}$$

Therefore the equations of motion are now more complicated, as expected, owing to the presence of charged particles.

Our problem now is to find a Hamiltonian function that gives a complete description of the system. That is, from it we should obtain the correct equations of motion, for the particles and for the field, using Hamilton's equations. A plausible form for this Hamiltonian, suggested by comparison of the left-hand sides of equations (3.13), (3.14), and (3.15) with the results of Section 2, would be a sum of terms of the form $\frac{1}{2}(P^2 + \omega^2 Q^2)$ of transverse, longitudinal, and scalar origin. The right-hand sides would suggest a con-

tribution from the Hamiltonian (3.1). We take as the nonrelativistic Hamiltonian

$$H = \sum_k \left\{ \frac{1}{2m_k} \left[\mathbf{p}_k - \frac{e_k}{c} \mathbf{A}(\mathbf{r}_k, t) \right]^2 + e_k \phi(\mathbf{r}_k, t) \right\} + \tfrac{1}{2} \sum_\lambda (P_\lambda^2 + \omega_\lambda^2 Q_\lambda^2) + \tfrac{1}{2} \sum_\sigma (P_\sigma^2 + \omega_\sigma^2 Q_\sigma^2) - \tfrac{1}{2} \sum_\sigma (P_{0\sigma}^2 + \omega_\sigma^2 Q_{0\sigma}^2) \tag{3.17}$$

As mentioned above, the justification for this Hamiltonian is that it works.

Unlike in the Lorentz gauge where $\mathbf{A}$ and ϕ satisfy (3.4) and (3.3) it is found that in the Coulomb gauge (2.1) the vector potential satisfies

$$\nabla^2 \mathbf{A}^T - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}^T}{\partial t^2} = -\frac{4\pi}{c} \mathbf{j}^T \tag{3.18}$$

and the scalar potential ϕ satisfies, at each instant, Poisson's equation

$$\nabla^2 \phi(\mathbf{r}, t) = -4\pi \rho(\mathbf{r}, t) \tag{3.19}$$

The vector potential is purely transverse since from div $\mathbf{A}^L = 0$ and curl $\mathbf{A}^L = 0$ we may write $\mathbf{A}^L = 0$. The vector $\mathbf{j}^T$ is the transverse component of the current density.

The Coulomb gauge has the disadvantage compared with the Lorentz gauge of being noncovariant, but it is more convenient to study the radiation field.

The solution for the equation (3.19) is

$$\phi(\mathbf{r}, t) = \int \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' \tag{3.20}$$

So, in this gauge the longitudinal part of $\mathbf{A}$ has been entirely eliminated from the equations of motion and the scalar potential reduces to the instantaneous Coulomb interaction of the charges.

In the Coulomb gauge the Hamiltonian (3.17) can be written

$$H = \sum_k \frac{1}{2m_k} \left[\mathbf{p}_k - \frac{e_k}{c} \mathbf{A}(\mathbf{r}_k, t) \right]^2 + \frac{1}{2} \sum_\lambda (P_\lambda^2 + \omega_\lambda^2 Q_\lambda^2) + V_{\text{coul}} \tag{3.21}$$

with

$$\mathbf{A}^T = \sum_\lambda Q_\lambda \mathbf{A}_\lambda \tag{3.22}$$

and using the Dirac delta function

$$V_{\text{coul}} = \frac{1}{2} \int \rho(\mathbf{r}, t) \phi(\mathbf{r}, t) \, d^3\mathbf{r} = \frac{1}{2} \sum_{i \neq k} \frac{e_i e_k}{|\mathbf{r}_i - \mathbf{r}_k|} \tag{3.23}$$

where it is understood that $i = k$ terms, representing the infinite self-energy, are omitted. The insufficient hypothesis we made about the charged particles makes the present calculations unsuitable for the discussion of self-energy problems.

Now we might endeavor to give an interpretation of the Hamiltonian (3.21) in terms of energy. The first term corresponds to the energy of the particles including the interaction energy between the charges and the radiation. The second term corresponds to the energy contained in the radiation field in the absence of sources. The third term is the static Coulomb interaction energy between the charges.

To include an external potential (ϕ^e, $\mathbf{A}^e$) we insert $\sum_k e_k \phi_k^e$ in (3.21) and consider in (3.22)

$$\mathbf{A} = \sum_{\lambda} Q_\lambda \mathbf{A}_\lambda + \mathbf{A}^e$$

4. QUANTIZATION OF THE FREE RADIATION FIELD

Once we have formally reduced the radiation field to an assembly of one-dimensional harmonic oscillators, the quantization is straightforward.

In the Heisenberg picture, the transition to quantum mechanics for a system having a classical analog can be achieved replacing the canonical variables $Q_{\mathbf{k}\sigma}$ and $P_{\mathbf{k}\sigma}$ by operators, changing in time, and satisfying the commutation relations

$$\begin{aligned} [P_{\mathbf{k}\sigma}, Q_{\mathbf{k}'\sigma'}] &= -i\hbar\delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'} \\ [Q_{\mathbf{k}\sigma}, Q_{\mathbf{k}'\sigma'}] &= [P_{\mathbf{k}\sigma}, P_{\mathbf{k}'\sigma'}] = 0 \end{aligned} \tag{4.1}$$

This procedure is based on the correspondence principle.

However, it is found that two non-Hermitian operators, corresponding to $a_{\mathbf{k}\sigma}$ and $a^*_{\mathbf{k}\sigma}$ (Section 2), and introduced by operator relations

$$a_{\mathbf{k}\sigma} = \left(\frac{1}{2\hbar\omega_k}\right)^{1/2} (\omega_k Q_{\mathbf{k}\sigma} + iP_{\mathbf{k}\sigma}) \tag{4.2}$$

and

$$a^\dagger_{\mathbf{k}\sigma} = \left(\frac{1}{2\hbar\omega_k}\right)^{1/2} (\omega_k Q_{\mathbf{k}\sigma} - iP_{\mathbf{k}\sigma}) \tag{4.3}$$

are more convenient. Their commutation rules follow from (4.1)

$$\begin{aligned} [a_{\mathbf{k}\sigma}, a^\dagger_{\mathbf{k}'\sigma'}] &= \delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'} \\ [a_{\mathbf{k}\sigma}, a_{\mathbf{k}'\sigma'}] &= [a^\dagger_{\mathbf{k}\sigma}, a^\dagger_{\mathbf{k}'\sigma'}] = 0 \end{aligned} \tag{4.4}$$

From (2.9) the Hamiltonian operator for the radiation field, without the zero-point energy, reduces to

$$H = \sum_{\mathbf{k}} \sum_{\sigma=1,2} \hbar\omega_k a^\dagger_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma} \tag{4.5}$$

For the purposes with which we are concerned, the infinite zero-point energy of the radiation field, $\sum_{\mathbf{k}\sigma} \hbar\omega_k/2$, corresponds to a constant term, which cancels out, being therefore unobservable.

From the Heisenberg equations of motion for $a_{\mathbf{k}\sigma}(t)$

$$\frac{da_{\mathbf{k}\sigma}}{dt} = \frac{1}{i\hbar}[a_{\mathbf{k}\sigma}(t), H] = -i\omega_k a_{\mathbf{k}\sigma}(t)$$

we obtain

$$a_{\mathbf{k}\sigma}(t) = a_{\mathbf{k}\sigma}(0)e^{-i\omega_k t} \tag{4.6}$$

and identically

$$a^\dagger_{\mathbf{k}\sigma}(t) = a^\dagger_{\mathbf{k}\sigma}(0)e^{i\omega_k t} \tag{4.7}$$

The operators $a_{\mathbf{k}\sigma}(0)$ and $a^\dagger_{\mathbf{k}\sigma}(0)$ correspond to the Schrödinger picture.

The Hermitian operator

$$N_{\mathbf{k}\sigma} = a^\dagger_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma} \tag{4.8}$$

is the number operator with eigenvalues $n_{\mathbf{k}\sigma} = 0, 1, 2, \ldots$ corresponding to the number of vibrational quanta, in the state $\mathbf{k}\sigma$, present in the cube. The eigenvectors $|n_{\mathbf{k}\sigma}\rangle$ of the observable $N_{\mathbf{k}\sigma}$ are the complete orthonormal basis of the representation $\{N_{\mathbf{k}\sigma}\}$.

Apart from a trivial phase

$$\begin{aligned} a^\dagger_{\mathbf{k}_i\sigma_i}|\ldots, n_{\mathbf{k}_i\sigma_i}, \ldots\rangle &= (n_{\mathbf{k}_i\sigma_i} + 1)^{1/2}|\ldots, n_{\mathbf{k}_i\sigma_i} + 1, \ldots\rangle \\ a_{\mathbf{k}_i\sigma_i}|\ldots, n_{\mathbf{k}_i\sigma_i}, \ldots\rangle &= n^{1/2}_{\mathbf{k}_i\sigma_i}|\ldots, n_{\mathbf{k}_i\sigma_i} - 1, \ldots\rangle \end{aligned} \tag{4.9}$$

and

$$|n_1, n_2, n_3, \ldots\rangle = \prod_\lambda \prod \frac{(a_\lambda^\dagger)^{n_\lambda}}{(n_\lambda!)^{1/2}}|0, 0, 0, \ldots\rangle \tag{4.10}$$

This discussion enables us to introduce a photon picture of the electromagnetic field. The state vector for the radiation field, $|n_1, n_2, \ldots, n_i, \ldots\rangle$, does not depend on time, since we found the Heisenberg picture to be a convenient representation. So, the time dependence is thrown into the field operators.

The eigenvalue problem for the Hamiltonian (4.5) is closely related to the eigenvalue problem for $N_{\mathbf{k}\sigma}$

$$E = \sum_{\mathbf{k}} \sum_{\sigma=1,2} n_{\mathbf{k}\sigma}\hbar\omega_k \tag{4.11}$$

Changing the complex coefficients (numbers) in (2.5) into operators $a^{\dagger}_{\mathbf{k}\sigma}$ and $a_{\mathbf{k}\sigma}$, which satisfy the commutation relations (4.4), we go from the classic to the quantic, throwing the function $\mathbf{A}(\mathbf{r}, t)$ into an operator, dependent on the parameters $\mathbf{r}$ and t, which acts on vectors of the N representation. This vector field operator is in the Heisenberg picture

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\mathbf{k}} \sum_{\sigma=1,2} \left(\frac{2\pi\hbar c^2}{\omega_k L^3}\right)^{1/2} [a_{\mathbf{k}\sigma}(t)\hat{e}_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}} + a^{\dagger}_{\mathbf{k}\sigma}(t)\hat{e}_{\mathbf{k}\sigma}e^{-i\mathbf{k}\cdot\mathbf{r}}] \quad (4.12)$$

Attending to (2.7), (2.8) and (2.9) it is straightforward to see that the normalization factor has been chosen such that $\mathbf{A}$ corresponds to the energy $\hbar\omega$ of one photon in the volume L^3 of the field.

The vector potential operator in the Schrödinger picture results from (4.12) at, say, $t = 0$.

It is of interest to point out that the number operator, $N_{\mathbf{k}\sigma}$, does not commute with the operators $\mathbf{A}(\mathbf{r}, t)$, $\mathscr{E}(\mathbf{r}, t)$ and $\mathscr{H}(\mathbf{r}, t)$.

Using for the momentum operator of the radiation field an operator expression identical to the classical

$$\mathbf{P} = \frac{1}{4\pi c}\int_{\text{cube}} \mathscr{E} \times \mathscr{H}\, d^3\mathbf{r} \quad (4.13)$$

and symmetrizing to ensure the Hermitian form, since $\mathscr{E}$ and $\mathscr{H}$ do not commute, we find

$$\mathbf{P} = \sum_{\mathbf{k}} \sum_{\sigma=1,2} \hbar\mathbf{k} a^{\dagger}_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma} \quad (4.14)$$

We have dropped a term $\sum_{\mathbf{k}\sigma} \hbar\mathbf{k}/2$ since the sum is carried out symmetrically on $\mathbf{k}$. Therefore, the states of (4.10) are also eigenvectors of the radiation momentum operator.

Associated with a vector field there is an angular momentum operator $\mathbf{J} = \mathbf{L} + \mathbf{S}$ which is the generator of its rotations. It might be shown (de Shalit and Talmi, 1963) that the expectation value of J_z for the electromagnetic field corresponds to the z component of the radiation angular momentum as classically defined by

$$\mathbf{J} = \frac{1}{4\pi c}\int \mathbf{r} \times (\mathscr{E} \times \mathscr{H})\, d^3\mathbf{r} \quad (4.15)$$

It can be shown (Gottfried, 1966) that the radiation angular momentum (4.15), as given in classical theory, can be written as

$$\mathbf{J} = \mathbf{L} + \mathbf{S} \quad (4.16)$$

where

$$\mathbf{L} = \frac{1}{4\pi c}\sum_{i=1}^{3}\int \mathscr{E}_i \mathbf{r} \times \nabla A_i\, d^3\mathbf{r}$$

and

$$\mathbf{S} = \frac{1}{4\pi c}\int \mathscr{E} \times \mathbf{A}\, d^3\mathbf{r}$$

Writing these expressions in Hermitian form and passing to quantum mechanics by changing to the corresponding operators, we obtain operators **L** and **S** that might be interpreted as representing, respectively, the orbital angular momentum and the spin of the radiation field. For **S** we find (Gottfried, 1966)

$$\mathbf{S} = -i\hbar \sum_{\mathbf{k}} \hat{\mathbf{k}}(a^\dagger_{\mathbf{k}1}a_{\mathbf{k}2} - a^\dagger_{\mathbf{k}2}a_{\mathbf{k}1}) \tag{4.17}$$

However, this operator does not commute with H. This difficulty may be overcome by passing from the linear polarization representation to the circular polarization representation

$$\hat{\xi}_{\mathbf{k}+1} = -2^{-1/2}(\hat{e}_{\mathbf{k}1} + i\hat{e}_{\mathbf{k}2})$$
$$\hat{\xi}_{\mathbf{k}-1} = 2^{-1/2}(\hat{e}_{\mathbf{k}1} - i\hat{e}_{\mathbf{k}2})$$

It is then possible to introduce new operators $a_{\mathbf{k}+1}$ and $a_{\mathbf{k}-1}$ defined by

$$\sum_{\sigma=1,2} \hat{e}_{\mathbf{k}\sigma}a_{\mathbf{k}\sigma} = \sum_{\lambda=\pm1} \hat{\xi}_{\mathbf{k}\lambda}a_{\mathbf{k}\lambda} \tag{4.18}$$

The new annihilation and creation operators satisfy commutation rules similar to (4.4) and we may write

$$\mathbf{S} = \sum_{\mathbf{k}} \sum_{\lambda=\pm1} \lambda\hbar\hat{\mathbf{k}}a^\dagger_{\mathbf{k}\lambda}a_{\mathbf{k}\lambda} \tag{4.19}$$

$$N_{\mathbf{k}\lambda} = a^\dagger_{\mathbf{k}\lambda}a_{\mathbf{k}\lambda} \tag{4.20}$$

$$H = \sum_{\mathbf{k}} \sum_{\lambda=\pm1} \hbar\omega_k a^\dagger_{\mathbf{k}\lambda}a_{\mathbf{k}\lambda} \tag{4.21}$$

$$\mathbf{P} = \sum_{\mathbf{k}} \sum_{\lambda=\pm1} \hbar\mathbf{k}a^\dagger_{\mathbf{k}\lambda}a_{\mathbf{k}\lambda} \tag{4.22}$$

These operators commute with each other and so it is possible to specify simultaneously their eigenvalues. Corresponding to the circular polarization unit vectors $\hat{\xi}_{\mathbf{k}\lambda}$ the operators $a_{\mathbf{k}\lambda}$ and $a^\dagger_{\mathbf{k}\lambda}$ correspond to the annihilation and creation of right-handed ($\lambda = +1$) or left-handed ($\lambda = -1$) circularly polarized photons.

Let us consider now a one-photon state defined by

$$|\mathbf{k}\lambda\rangle = a^\dagger_{\mathbf{k}\lambda}|0\rangle, \qquad \lambda = \pm 1 \tag{4.23}$$

It can be shown that for this state

$$(\mathbf{L}\cdot\hat{\mathbf{k}})|\mathbf{k}\lambda\rangle = 0$$

This is not a surprising result for an orbital angular momentum acting on a state described by a plane wave.

From (4.19) it follows that

$$(\mathbf{S}\cdot\hat{\mathbf{k}})|\mathbf{k}\lambda\rangle = \lambda\hbar|\mathbf{k}\lambda\rangle, \qquad \lambda = \pm 1$$

It is of interest to point out, at this stage, that as a result of the quantization procedure we were led to the photon picture of the radiation field. The occupation states represent a number of particles, i.e., states associated with some event and corresponding to a set of quantum numbers. We may interpret the result (4.19) as describing the spin of the radiation field in terms of a total number of photons each of them described in terms of helicity $+1$ and -1.

For the component of the total angular momentum of the radiation field along the direction of propagation, and for a one-photon state we have

$$J_k|\mathbf{k}\lambda\rangle = \lambda\hbar|\mathbf{k}\lambda\rangle, \qquad \lambda = \pm 1 \tag{4.24}$$

We say that the photon helicity, defined as the projection of the total angular momentum of the photon along the direction of propagation, can only have the values $+1$ and -1. The value zero is not possible. This is a property characteristic of spin-1 particles with zero rest mass, and it leads to an association between the quantum of the radiation field and a particle of mass 0 and spin 1.

Using the operators introduced in (4.18) we can easily extend the transverse vector field operator $\mathbf{A}(\mathbf{r}, t)$ (4.12)

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\mathbf{k}} \sum_{\lambda=\pm 1} \left(\frac{2\pi\hbar c^2}{\omega_k L^3}\right)^{1/2} [a_{\mathbf{k}\lambda}\hat{\xi}_{\mathbf{k}\lambda} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + a^{\dagger}_{\mathbf{k}\lambda}\hat{\xi}^{*}_{\mathbf{k}\lambda} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}] \tag{4.25}$$

The set $|\{n_{\mathbf{k}\lambda}\}\rangle$ is very convenient to use in cases involving perturbation theory calculation with variation of the number of photons by 1. It does not follow that this is the only representation possible or even that it always offers the best insight in radiation problems. The coherent states (Glauber, 1963; Carruthers and Nieto, 1965) are very useful in certain radiation problems because they are the quantum states that more closely approximate the classical limit. So far we have used the number representation, but here the expectation values of $\mathscr{E}$, $\mathscr{H}$, and $\mathbf{A}$ vanish

$$\langle n_1, n_2, \ldots, n_\lambda, \ldots|\mathscr{E}|n_1, n_2, \ldots, n_\lambda, \ldots\rangle = 0$$

since these operators have no diagonal matrix elements in that representation. But the classical description of a system should be valid as obtained from the

quantum description when the quantum numbers are huge compared to unity. So the correspondence principle cannot be applied here because no matter how large the n's are we do not obtain results valid in the classical limit. It is, nevertheless, possible to construct states that are the quantum analogs of the classical, by superposition, in an appropriate way, of energy eigenstates of the harmonic oscillator. This coherent state $|\alpha\rangle$ may be defined as an eigenstate of the operator

$$a|\alpha\rangle = \alpha|\alpha\rangle$$

For the operator $a^\dagger$ no such relation exists because it would imply $\langle n|\alpha\rangle = 0$ for any n.

The coherent state, defined for all complex numbers α, forms a complete nonorthogonal set (a is a non-Hermitian operator)

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{(n!)^{1/2}} |n\rangle = e^{-|\alpha|^2/2} e^{\alpha a^\dagger} |0\rangle$$

where $|0\rangle$ is the ground state of the oscillator and $e^{-|\alpha|^2/2}$ a normalization factor.

The probability that the energy of the coherent state is $n\hbar\omega$ follows a Poisson distribution

$$|\langle n|\alpha\rangle|^2 = \frac{(|\alpha|^2)^n}{n!} e^{-|\alpha|^2/2}$$

The coherent state represents a minimum uncertainty wave packet state because it minimizes the product $\Delta q \Delta p$.

The expectation values of a and $a^\dagger$ in this state are

$$\langle\alpha|a|\alpha\rangle = \alpha$$

$$\langle\alpha|a^\dagger|\alpha\rangle = \alpha^*$$

For a multimode state

$$|\alpha_1, \alpha_2, \ldots, \alpha_\lambda, \ldots\rangle = \prod_\lambda e^{-|\alpha_\lambda|^2/2} e^{\alpha_\lambda a_\lambda^\dagger} |0\rangle$$

with

$$a_\lambda|\alpha_1, \alpha_2, \ldots, \alpha_\lambda, \ldots\rangle = \alpha_\lambda|\alpha_1, \alpha_2, \ldots, \alpha_\lambda, \ldots\rangle$$

If each mode is in a coherent state the mean value of, say, $\mathscr{E}(\mathbf{r}, t)$ is

$$\langle\alpha_1, \ldots, \alpha_\lambda, \ldots|\mathscr{E}(\mathbf{r}, t)|\alpha_1, \ldots, \alpha_\lambda, \ldots\rangle$$

$$= \frac{i}{c} \sum_\lambda \left(\frac{2\pi\hbar c^2}{\omega_\lambda L^3}\right)^{1/2} \omega_\lambda [\alpha_\lambda e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_\lambda t)} - \alpha_\lambda^* e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega_\lambda t)}]$$

and identically for $\mathscr{H}(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$, looking like the classical field (2.7).

5. THE INTERACTION HAMILTONIAN BETWEEN THE RADIATION FIELD AND NONRELATIVISTIC CHARGES

In view of the result (3.21) the nonrelativistic Hamiltonian for a system of particles (without spin) plus radiation field may be written, in the Schrödinger picture,

$$H = \sum_k \frac{1}{2m_k}\left[\mathbf{p}_k - \frac{e_k}{c}\mathbf{A}(\mathbf{r}_k)\right]^2 + H_{\text{rad}} + V(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_k, \ldots) \tag{5.1}$$

where V is the interaction potential between the particles and H_{rad} can be written in the form (4.21).

Writing

$$H_{\text{part}} = \sum_k \frac{1}{2m_k}\mathbf{p}_k^2 + V \tag{5.2}$$

the Hamiltonian (5.1) can be written as

$$H = H_{\text{part}} + H_{\text{rad}} + H_{\text{int}} \tag{5.3}$$

The term

$$H_{\text{int}} = -\sum_k \frac{e_k}{2m_k c}(\mathbf{p}_k\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{p}_k) + \sum_k \frac{e_k^2}{2m_k c^2}\mathbf{A}^2 \tag{5.4}$$

describing the interaction between the field and particles, is the perturbing energy on the unperturbed Hamiltonian

$$H_0 = H_{\text{part}} + H_{\text{rad}} \tag{5.5}$$

whose eigenfunction can be written in the form

$$|\Psi\rangle = |i\rangle_{\text{part}}|\ldots, n_{\mathbf{k}\lambda}, \ldots\rangle_{\text{rad}} \tag{5.6}$$

The second term of H_{int} (5.4) gives rise to products of the form $a_{\mathbf{k}\lambda}a_{\mathbf{k}'\lambda'}$, $a_{\mathbf{k}\lambda}a^\dagger_{\mathbf{k}'\lambda'}$, $a^\dagger_{\mathbf{k}\lambda}a_{\mathbf{k}'\lambda'}$, and $a^\dagger_{\mathbf{k}\lambda}a^\dagger_{\mathbf{k}'\lambda'}$ and must be considered in processes involving two photons. Otherwise this term is of second order in e and can be neglected in perturbation calculations taken to first order in e. In such a case we may take simply

$$H_{\text{int}} = -\sum_k \frac{e_k}{2m_k c}(\mathbf{p}_k\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{p}_k) \tag{5.7}$$

In fact, since

$$[p_j, f(\mathbf{q}, \mathbf{p})] = -i\hbar\frac{\partial f}{\partial q_j} \tag{5.8}$$

with $j(1, 2, 3)$ referring to components, we may write in the Coulomb gauge $\mathbf{p}\cdot\mathbf{A} = \mathbf{A}\cdot\mathbf{p}$.

Considering a quantum mechanical system of particles with masses m_k and charges e_k, the charge-density operator is given by a sum of δ functions. For a system of protons and neutrons

$$\rho_{\text{op}}(\mathbf{r}) = \sum_k e g_l(k)\delta(\mathbf{r} - \mathbf{r}_k) \tag{5.9}$$

where $g_l(k) = 1$ or 0 for a proton or a neutron, $\mathbf{r}_k$ is a position operator and $\mathbf{r}$, which is not an operator, indicates the point of space where we observe the charge density.

The charge density in a state is given by the expectation value of (5.9) in that state. This expectation value corresponds to the sum of the probabilities of finding each of the charges, and for all possible configurations, at the position denoted by $\mathbf{r}$.

Similarly we may define a current-density operator

$$\mathbf{j}_{\text{op}}(\mathbf{r}) = \sum_k \frac{e g_l(k)}{2m} [\mathbf{p}_k\delta(\mathbf{r} - \mathbf{r}_k) + \delta(\mathbf{r} - \mathbf{r}_k)\mathbf{p}_k] \tag{5.10}$$

It is convenient to symmetrize since the expression contains the product of noncommuting Hermitian operators and we want to ensure that $\mathbf{j}_{\text{op}}(\mathbf{r})$ is Hermitian. For the simple case of a single particle it can be shown, using (5.8) and the property of the δ function $f(x)\delta'(x - a) = -f'(x)\delta(x - a)$, that

$$\int \varphi^*(\mathbf{r}_k)\mathbf{j}_{\text{op}}(\mathbf{r})\varphi(\mathbf{r}_k)\, d^3\mathbf{r}_k = -\frac{ie\hbar}{2m} [\varphi^*(\mathbf{r})\nabla\varphi(\mathbf{r}) - \varphi(\mathbf{r})\nabla\varphi^*(\mathbf{r})] \tag{5.11}$$

We might say that this gives the idea of introducing a particle current-density operator (5.10) whose expectation value is the probability density current shown on the right side of (5.11).

In terms of the current-density operator $\mathbf{j}_{\text{op}}(\mathbf{r})$ we may write H_{int} (5.7), in the Schrödinger picture, as

$$\begin{aligned} H^S_{\text{int}} &= -\sum_k \int d^3\mathbf{r}\, \frac{e g_l(k)}{2mc} [\mathbf{p}_k\cdot\delta(\mathbf{r} - \mathbf{r}_k)\mathbf{A}(\mathbf{r}) + \delta(\mathbf{r} - \mathbf{r}_k)\mathbf{A}(\mathbf{r})\cdot\mathbf{p}_k] \\ &= -\frac{1}{c}\int \mathbf{j}_{\text{op}}(\mathbf{r})\cdot\mathbf{A}(\mathbf{r})\, d^3\mathbf{r} \end{aligned} \tag{5.12}$$

describing the coupling of the quantized radiation field to the current density of the particle system.

In the following we will omit the subscript "op."

6. THE NUCLEUS AS A SYSTEM OF NONRELATIVISTIC SPIN-$\frac{1}{2}$ PARTICLES

Just as classically, a spinning charge distribution corresponds to a magnetic moment, we associated in quantum mechanics a magnetic moment with the spin. In the nucleus the total current operator has a contribution from the "convection" current and a contribution from the "spin" current coming from the intrinsic magnetic moment of the nucleons. A magnetic moment density $\boldsymbol{\mu}$ is equivalent to a current (Jackson, 1962)

$$\mathbf{j}_M = c\nabla \times \boldsymbol{\mu} \tag{6.1}$$

For a single particle we define a density operator corresponding to the intrinsic angular momentum or spin

$$\mathbf{s}(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r}_k)\tfrac{1}{2}\hbar\boldsymbol{\sigma} \tag{6.2}$$

where $\boldsymbol{\sigma}(\sigma_x, \sigma_y, \sigma_z)$ is the operator corresponding to the Pauli matrices acting on the spin part of a two-component wave function called a spinor. The expectation value of $\mathbf{s}(\mathbf{r})$ is the spin density at the position $\mathbf{r}$.

The magnetic moment of spin density operator for a point charge is taken as

$$\boldsymbol{\mu}(\mathbf{r}) = \frac{e}{2mc} g_s(k)\delta(\mathbf{r} - \mathbf{r}_k)\frac{\hbar}{2}\boldsymbol{\sigma} \tag{6.3}$$

where g_s is the spin g factor, with $g_s = 5.585$ for a proton and $g_s = -3.826$ for a neutron.

For a many-particle system

$$\boldsymbol{\mu}(\mathbf{r}) = \sum_k \frac{e}{2mc} g_s(k)\delta(\mathbf{r} - \mathbf{r}_k)\frac{\hbar}{2}\boldsymbol{\sigma}(k) \tag{6.4}$$

where $\boldsymbol{\sigma}(k)$ is the operator acting on the spin coordinates of the kth particle. Calculating the expectation value we integrate for, every k, over the coordinates of the other particles.

The total current density operator is

$$\mathbf{J}(\mathbf{r}) = \mathbf{j}(\mathbf{r}) + c\nabla \times \boldsymbol{\mu}(\mathbf{r}) \tag{6.5}$$

The interaction between the total nuclear current and the quantized radiation field, in the Schrödinger picture, is

$$H_{\text{int}}^S = -\frac{1}{c}\int \mathbf{J}(\mathbf{r})\cdot\mathbf{A}(\mathbf{r})\, d^3\mathbf{r} \tag{6.6}$$

Once $\mathbf{A}(\mathbf{r})$, in the Schrödinger picture, results from (4.25) at, say, $t = 0$, this operator only has matrix elements for transitions in which the number of photons change by 1.

From the previous considerations as $\nabla \times \mu \cdot \mathbf{A} = \mu \cdot \mathscr{H}$, the contribution to the interaction Hamiltonian of the current resulting from the intrinsic magnetization can be interpreted as the energy of interaction of the magnetic moment with the magnetic field.

7. EMISSION OF ELECTROMAGNETIC RADIATION

In accordance with (5.3) and (5.5) we may write, in the Schrödinger picture,

$$H^S = H_0{}^S + H_{\text{int}}^S \tag{7.1}$$

$H_0{}^S$ contains the description of the two systems, radiating system and radiation field, as noninteracting, and H_{int}^S describes their mutual interaction. It is this interaction, of the nucleus with the external field, which causes transitions to take place between stationary states of the system. With H_N the Hamiltonian of the radiating system (nucleus), we may write for the time independent Hamiltonian $H_0{}^S$

$$H_0{}^S = H_N + H_{\text{rad}} \tag{7.2}$$

with eigenfunctions

$$|\psi_n{}^S\rangle = |i\rangle_N |\cdots n_{\mathbf{k}\lambda}\cdots\rangle_{\text{rad}} \tag{7.3}$$

A convenient way to deal with the problem is to transform the Schrödinger state

$$|\psi^S(t)\rangle = \sum_n c_n(t) e^{-(i/\hbar)E_n{}^0 t} |\psi_n{}^S\rangle \tag{7.4}$$

solution of

$$i\hbar \frac{\partial |\psi^S(t)\rangle}{\partial t} = H^S |\psi^S(t)\rangle$$

into the interaction picture.

With the superscript I denoting interaction picture we obtain for the state vectors and perturbation, respectively

$$|\psi^I(t)\rangle = e^{(i/\hbar)H_0 t} |\psi^S(t)\rangle = \sum_n c_n(t) |\psi_n{}^S\rangle \tag{7.5}$$

and

$$H_{\text{int}}^I = e^{(i/\hbar)H_0 t} H_{\text{int}}^S e^{-(i/\hbar)H_0 t} \tag{7.6}$$

In the interaction picture we obtain

$$i\hbar \frac{\partial |\psi^I(t)\rangle}{\partial t} = H_{\text{int}}^I |\psi^I(t)\rangle \tag{7.7}$$

where H^I_{int} appears instead of H^S, showing that the time development of $|\psi^I(t)\rangle$ is dictated by the interaction Hamiltonian.

It would be interesting to note, in addition to what we have said in Section 3, that the interaction picture is implied in the process of "disregarding the external world" from our problems.

The equations of motion for the coefficients $c_n(t)$ are obtained from (7.7)

$$\frac{dc_m}{dt} = -\frac{i}{\hbar}\sum_n c_n(t)\langle\psi_m{}^S|H^I_{\text{int}}|\psi_n{}^S\rangle \tag{7.8}$$

Let us now consider a nucleus that emits a photon of momentum $\hbar\mathbf{k}$ and helicity λ, making a transition from the initial nuclear state $|i\rangle$ to the final state $|f\rangle$ belonging to a discrete spectrum. Supposing that the system is initially in an eigenstate of H_0, (7.2), we may refer to this eigenstate as $|i;\cdots n_{\mathbf{k}\lambda}\cdots\rangle$ representing the product of the nuclear state $|i\rangle$ by the radiation state $|\cdots n_{\mathbf{k}\lambda}\cdots\rangle$. The only final states for which there is a nonvanishing matrix element of H_{int} are states $|f;\cdots n_{\mathbf{k}\lambda}+1\cdots\rangle$.

We may write for the matrix element involved in (7.8)

$$\begin{aligned}
&\langle f;\cdots n_{\mathbf{k}\lambda}+1\cdots|H^I_{\text{int}}|i;\cdots n_{\mathbf{k}\lambda}\cdots\rangle \\
&\quad=\left(\frac{2\pi\hbar}{\omega_k L^3}\right)^{1/2}\langle f;\cdots n_{\mathbf{k}\lambda}+1\cdots|\int d^3\mathbf{r}[e^{(i/\hbar)H_N t}\mathbf{J}(\mathbf{r})e^{-(i/\hbar)H_N t}] \\
&\qquad\cdot\hat{\xi}^*_{\mathbf{k}\lambda}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}a^\dagger_{\mathbf{k}\lambda}|i;\cdots n_{\mathbf{k}\lambda}\cdots\rangle \\
&\quad=e^{-(i/\hbar)(E_i-E_f-\hbar\omega_k)t}\langle f;\cdots n_{\mathbf{k}\lambda}+1\cdots|\mathscr{H}^\dagger_{\text{int}}|i;\cdots n_{\mathbf{k}\lambda}\cdots\rangle
\end{aligned} \tag{7.9}$$

with

$$\mathscr{H}^\dagger_{\text{int}} = -\left(\frac{2\pi\hbar}{\omega_k L^3}\right)^{1/2}\int d^3\mathbf{r}\mathbf{J}(\mathbf{r})\cdot\hat{\xi}^*_{\mathbf{k}\lambda}e^{-i\mathbf{k}\cdot\mathbf{r}}a^\dagger_{\mathbf{k}\lambda} \tag{7.10}$$

Only the term in $a^\dagger_{\mathbf{k}\lambda}$ is present in the matrix element because the corresponding term in $a_{\mathbf{k}\lambda}$ cannot contribute. With the final state written on the left, as in (7.9), the operator adequate to the emission matrix element is the operator $\mathscr{H}^\dagger_{\text{int}}$.

Actually, the emission involves a continuum of photon states. Until now we have been assuming that the nucleus is heavy, localized, and the levels infinitely sharp, with no linewidth, and so perfectly monochromatic radiation. In fact this implies neglecting the reaction of the radiation field on the radiating system, which is not realistic. Perhaps it would be useful to go back to the approach used in Section 2 to treat the continuous problem. To substitute it by a discrete problem we confined the field in a cube of volume L^3. After taking advantage of this substitution we keep always in mind that to go back

to the initial problem we just need to make L increase indefinitely. As we have seen in Section 2, the values for $\mathbf{k}$, allowed by the periodic boundary conditions, are $\mathbf{k} = (2\pi/L)\mathbf{n}$. When L increases, the allowable $\mathbf{k}$'s approximate a continuous distribution in $\mathbf{k}$ space and the photon states approximate a continuous energy spectrum. As we are interested, physically, in studying the emission of a photon within a solid angle $d\Omega$, i.e., over a small range of $\mathbf{k}$, we may write for the number of photon states, of one helicity λ, with energy between $E = \hbar\omega$ and $E + dE$, and in a solid angle $d\Omega$ about $\mathbf{k}$

$$\rho_{d\Omega}(E)\, dE = \frac{L^3}{(2\pi c)^3}\frac{\omega^2}{\hbar}\, d\Omega\, dE \tag{7.11}$$

where $\rho_{d\Omega}(E)$ is called the density of states.

On the other hand, the solution of (7.8) depends critically on the initial conditions. The standard treatment of first-order time-dependent perturbation theory, leading to Fermi's golden rule, is restricted to times short enough to make the probability of transition out of the initial state very small. That is, $c_i(t)$ may be taken as equal to $c_i(0) = 1$ minus small first-order corrections. Obviously these are not the conditions we expect to be satisfied when studying the decay of a state. So, we will resort to the Weisskopf–Wigner method assuming an exponential decay law, $|c_i(t)|^2 = e^{-\Gamma t}$, for the initial state. The quantity $\tau = \Gamma^{-1}$ is the mean lifetime of the excited state. Neglecting the level shift we obtain the result

$$\Gamma = \sum_{\mathbf{k}}\sum_{\lambda}\frac{2\pi}{\hbar}\,|\langle f;\cdots n_{\mathbf{k}\lambda}+1\cdots|\mathscr{H}^{\dagger}_{\text{int}}|i;\cdots n_{\mathbf{k}\lambda}\cdots\rangle|^2\rho_{d\Omega}(E_i - E_f) \tag{7.12}$$

where it is understood we sum over the $\mathbf{k}$ in $d\Omega$, and $\rho_{d\Omega}(E_i - E_f)$ represents the density of states of emitted photons with energy $E = \hbar\omega = E_i - E_f$. Actually, this expression corresponds to the transition probability per second obtained in first-order time-dependent perturbation theory.

We may write for the matrix element involved in (7.12)

$$\langle f;\cdots n_{\mathbf{k}\lambda}+1\cdots|\mathscr{H}^{\dagger}_{\text{int}}|i;\cdots n_{\mathbf{k}\lambda}\cdots\rangle$$
$$= -\left(\frac{2\pi\hbar}{\omega_k L^3}\right)^{1/2}(n_{\mathbf{k}\lambda}+1)^{1/2}\int d^3\mathbf{r}\langle f|\mathbf{J}(\mathbf{r})|i\rangle\cdot\hat{\xi}^{*}_{\mathbf{k}\lambda}e^{-i\mathbf{k}\cdot\mathbf{r}} \tag{7.13}$$

The term in (7.13) proportional to $n_{\mathbf{k}\lambda}$ corresponds to the stimulated emission. The term that is still present even when $n_{\mathbf{k}\lambda} = 0$ corresponds to the spontaneous emission. This means that the matrix element (7.13) is not necessarily zero when there are no photons $\mathbf{k}\lambda$ already present in the initial state, and so this explains the emission of a photon by an isolated nucleus when there is no applied radiation.

The expression

$$e^{(i/\hbar)H_N t}\mathbf{J}(\mathbf{r})e^{-(i/\hbar)H_N t} = \mathbf{J}^H(\mathbf{r}, t) \tag{7.14}$$

appearing in (7.9), is the current density operator in the Heisenberg picture for an isolated nucleus. We can write a similar expression for $\rho^H(\mathbf{r}, t)$. In the following we shall omit the superscripts.

When we related the field with sources, in Section 3, we were assuming a *classical* localized distribution of charge $\rho(\mathbf{r}, t)$ and current $\mathbf{J}(\mathbf{r}, t)$ whose time dependence could be represented by a superposition of Fourier components. For our purposes it is sufficient to consider the part with frequency ω, $\rho(\mathbf{r}, t) = \rho(\mathbf{r})e^{-i\omega t}$ and $\mathbf{J}(\mathbf{r}, t) = \mathbf{J}(\mathbf{r})e^{-i\omega t}$ where it is assumed to take the real parts. We may say that passing to quantum mechanics the *classical* charge and current densities are replaced by the matrix elements of the corresponding *operators*, $\rho(\mathbf{r}, t)$ and $\mathbf{J}(\mathbf{r}, t)$, between the initial and final states of the radiating system. So they go into a transition current density

$$\langle f|\mathbf{J}(\mathbf{r}, t)|i\rangle = e^{-(i/\hbar)(E_i - E_f)t}\langle f|\mathbf{J}(\mathbf{r})|i\rangle \tag{7.15}$$

and identically for the charge density.

The above treatment, although good enough for some purposes, does not take into account the finite size of nucleons as well as meson effects such as exchange currents (Bohr and Mottelson, 1969).

The finite size of nucleons can be taken into account by a suitable substitution of the delta function, which appears in the charge-density and current-density operators, by convenient magnetic and electric form factors.

The presence of nuclear forces, supposed to be originated in the meson exchange between nucleons, generates exchange currents producing electromagnetic effects. Till now we only assumed, explicitly, the Coulomb forces between the charges. But, of course, for the nucleus the Hamiltonian is assumed to contain the exchange effects, with the nuclear state being eigenstate of the meson–nucleon problem.

Even with H containing exchange potentials, if $\mathbf{j}$ includes the exchange current $\mathbf{j}_{\text{exch}}$ defined in such a way that

$$\nabla\cdot\mathbf{j}_{\text{exch}} = -(i/\hbar)[V_{\text{exch}}, \rho]$$

$\mathbf{j}$ is still assumed to satisfy the charge conservation continuity equation (the spin current does not contribute since it has zero divergence)

$$\nabla\cdot\mathbf{j} = -\frac{\partial\rho}{\partial t} = -\frac{i}{\hbar}[H, \rho] \tag{7.16}$$

with H the Hamiltonian of the nucleus.

The difficulties in handling this problem result from assumptions we need to describe the meson current distribution.

Siegert's theorem (Sachs, 1953) can be invoked for the electric multipole moments to replace the current operator by the charge density operator. This theorem depends on the fact that, in the long-wavelength limit, **A** can be written approximately as the gradient of a scalar. For a magnetic multipole this cannot be done. So, the exchange interactions are expected, in good approximation, to have no important effects in the electric multipole moments, once the charge distribution can be assumed not to be much affected by the exchange currents. On the other hand, magnetic multipoles depend on the current density, Siegert's theorem does not apply, and the exchange effects can be important and are exhibited in large "anomalous" magnetic moments, indicating strong meson currents.

8. THE ANGULAR MOMENTUM AND PARITY OF THE PHOTON

Till now we have been considering the emission of photons with given momentum and helicity. However, this is not convenient for calculations involving emission of radiation between states with definite values of angular momentum and parity. In fact parity "violations" in nuclear states are so weak that they can be neglected. As the interaction Hamiltonian (6.6) is a scalar product of two vectors, it commutes with the operators which generate the infinitesimal rotations of coordinates and with the parity operation. In this way the angular momentum and parity are conserved in electromagnetic transitions between nuclear states and it would be convenient to attribute, as a particle, to the emitted photon definite values of angular momentum and parity.

We shall now make some comments on the physical significance of the description of the angular momentum of the photon as composed of an orbital part and intrinsic spin of unity.

The operator of annihilation of circularly polarized photons, $a_{\mathbf{k}\lambda}$, appears in H_{int} together with a plane wave $\hat{\xi}_{\mathbf{k}\lambda}e^{i\mathbf{k}\cdot\mathbf{r}}$, which satisfies the wave equation $(\nabla^2 + k^2)\mathbf{A}(\mathbf{r}) = 0$. We could take the photon wave function as corresponding, in coordinate space, to this function. If we adopt that function as describing the state of a single photon with definite momentum and helicity, we may write

$$\langle \mathbf{r}|\mathbf{k}\lambda\rangle = \hat{\xi}_{\mathbf{k}\lambda}e^{i\mathbf{k}\cdot\mathbf{r}} \tag{8.1}$$

The lack of meaning of a localized photon is implicit in this expression because it contains only the momentum in the direction normal to the plane wave, so it is not possible to localize the photon in that plane. The photon, represented by the vector state (8.1), may be said to have an angular momentum that in principle could be decomposed into an orbital angular momentum related to the change in spatial coordinates and an intrinsic spin part related

to rotations of the vector's direction in every point. So, even if we can mathematically separate these two operations, only the total angular moment of the photon has a physical meaning, because only the rotations associated to it correspond to physically significant results.

We shall see next that photon states with definite parity are linear combinations of states with $\lambda = +1$ and $\lambda = -1$ with equal intensities.

We can project out of the plane wave $|\mathbf{k}\lambda\rangle$, supposedly referred to the moving system, angular momentum states $|JM\lambda\rangle$ referred to the fixed frame $(\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z)$

$$|JM\lambda\rangle = \left(\frac{2J+1}{4\pi}\right)^{1/2} \int D^J_{M\lambda}(\hat{\mathbf{k}})|\mathbf{k}\lambda\rangle \, d\hat{\mathbf{k}} \tag{8.2}$$

By $\int d\hat{\mathbf{k}}$ we indicate integration over the unit sphere. We shall use for the D matrices the convention of Bohr and Mottelson (1969). The rotation of the $\hat{\mathbf{k}} \equiv \hat{\xi}_{\mathbf{k}0}$ axis of the moving system to the $\hat{\mathbf{e}}_z \equiv \hat{\xi}_0$ axis of the fixed frame is defined by the Euler angles $\alpha \equiv \varphi$ and $\beta \equiv \theta$, where α is the angle that makes the y axis perpendicular to the plane of $\hat{\mathbf{k}}$ and $\hat{\mathbf{e}}_z$. As γ is not constrained we may choose $\gamma = 0$. We may write for (8.2)

$$|JM\lambda\rangle = \left(\frac{2J+1}{4\pi}\right)^{1/2} \int D^J_{M\lambda}(\varphi, \theta, 0)|\mathbf{k}\lambda\rangle \sin\theta \, d\theta \, d\varphi \tag{8.3}$$

with the normalization factor coming from

$$\int D^{J*}_{M\lambda}(\varphi, \theta, 0) D^{J'}_{M'\lambda}(\varphi, \theta, 0) \sin\theta \, d\theta \, d\varphi = \frac{4\pi}{2J+1} \delta_{JJ'}\delta_{MM'}$$

To the state $|JM\lambda\rangle$ corresponds obviously the magnitude of momentum k because the wave vector $\mathbf{k}$ corresponds to (k, φ, θ) and the integration is over φ and θ. So we could insert it in the ket and write $|k, JM\lambda\rangle$ instead of $|JM\lambda\rangle$.

Let us now consider the behavior of $|k, JM\lambda\rangle$ under parity operation. To begin with

$$D^J_{M\lambda}(\pi + \varphi, \pi - \theta, 0) = (-)^J D^J_{M-\lambda}(\varphi, \theta, 0) \tag{8.4}$$

Then from (8.1) we shall have to examine what happens to $\hat{\xi}_{\mathbf{k}\lambda}$ under parity operation. To find the result we consider $\hat{\xi}_{\mathbf{k}\lambda}$ as written in its spherical components relative to the fixed frame

$$\hat{\xi}_{\mathbf{k}\lambda} = \sum_\nu \hat{\xi}_\nu D^{1*}_{\nu\lambda}(\varphi, \theta, 0)$$

Here $\hat{\xi}_\nu$ is given by

$$\hat{\xi}_{\pm 1} = \mp 2^{-1/2}(\hat{\mathbf{e}}_x \pm i\hat{\mathbf{e}}_y), \qquad \hat{\xi}_0 = \hat{\mathbf{e}}_z$$

From (8.4) we may write

$$\hat{\xi}_{-\mathbf{k}\lambda} = -\hat{\xi}_{\mathbf{k}-\lambda}$$

and finally

$$P\langle \mathbf{r}|\mathbf{k}\lambda\rangle = P\hat{\xi}_{\mathbf{k}\lambda}e^{i\mathbf{k}\cdot\mathbf{r}} = -\hat{\xi}_{\mathbf{k}-\lambda}e^{i\mathbf{k}\cdot\mathbf{r}} = -\langle \mathbf{r}|\mathbf{k}-\lambda\rangle \tag{8.5}$$

So, the result of the parity operation on (8.3) is

$$P|k, JM\lambda\rangle = (-)^J|k, JM-\lambda\rangle \tag{8.6}$$

Corresponding to the values kJM we may construct states with well defined parity. These states can be formed by linear combinations of states with $\lambda = \pm 1$

$$|k\pm; JM\rangle = 2^{-1/2}(|k; JM1\rangle \pm |k; JM-1\rangle) \tag{8.7}$$

From (8.6) the parity of these states is given by

$$P|k\pm; JM\rangle = \pm(-)^J|k\pm; JM\rangle \tag{8.8}$$

This behavior is closely related to the electric and magnetic multipole operators that we shall consider in the next section. The angular momentum J and parity $(-)^J$ of the photon $|k+; JM\rangle$ corresponds to an electric 2^J-pole transition and the angular momentum J and parity $(-)^{J+1}$ of the photon $|k-; JM\rangle$ to a magnetic 2^J-pole transition.

The photon states $|k\pm; JM\rangle$ are created by operators that we can define as

$$a^\dagger_{\pm JM} = 2^{-1/2}(a^\dagger_{JM1} \pm a^\dagger_{JM-1}) \tag{8.9}$$

where identically to (8.2)

$$a^\dagger_{JM\lambda} = \left(\frac{2J+1}{4\pi}\right)^{1/2}\int D^J_{M\lambda}(\hat{\mathbf{k}})a^\dagger_{\mathbf{k}\lambda}\,d\hat{\mathbf{k}} \tag{8.10}$$

9. THE MULTIPOLE OPERATORS

We shall now concentrate on the matrix element of the emission operator (7.13) corresponding to a transition from an initial nuclear state $|i\rangle$ into a final nuclear state $|f\rangle$

$$\int \langle f|\mathbf{J}(\mathbf{r})|i\rangle\cdot\hat{\xi}^*_{\mathbf{k}\lambda}e^{-i\mathbf{k}\cdot\mathbf{r}}\,d^3\mathbf{r} \tag{9.1}$$

The plane wave solutions of the wave equation represent well-defined momentum states but on the other hand do not represent well-defined angular momentum states. So, this description is not convenient for calculations involving emission (or absorption) of radiation between nuclear states. This suggests the multipole expansion of the vector field in terms of spherical

waves corresponding each to defined values of angular momentum and parity. The electric and magnetic multipole solutions of the wave equation $(\nabla^2 + k^2)\mathbf{A} = 0$ have properties appropriate for such an expansion. These electric and magnetic components of the transverse field are

$$\mathbf{A}^{(e)}_{JM}(\mathbf{r}) = \frac{i^J}{k} \frac{\nabla \times \mathbf{L}}{[J(J+1)]^{1/2}} j_J(kr) Y^{(\theta,\varphi)}_{JM} \tag{9.2}$$

with parity $(-)^{J+1}$, and

$$\mathbf{A}^{(m)}_{JM}(\mathbf{r}) = i^J \frac{\mathbf{L}}{[J(J+1)]^{1/2}} j_J(kr) Y^{(\theta,\varphi)}_{JM} \tag{9.3}$$

with parity $(-)^J$. The function $j_J(kr)$ is a spherical Bessel function. The factor i^J is used to introduce convenient time-reversal properties.

Because $\mathbf{L}$ applied to $Y_{00}(\theta, \varphi)$ gives zero, we may note that (9.2) and (9.3) rule out any transverse solution with $J = 0$.

We may expand the transverse circularly polarized plane wave in terms of the set formed by the divergenceless multipole fields (9.2) and (9.3) (Brink and Satchler, 1968).

For a coordinate system with the z axis along the wave vector $\mathbf{k}$ $(M = \lambda = \pm 1)$ we have

$$\hat{\xi}_{\mathbf{k}\lambda} e^{ikz} = -(2\pi)^{1/2} \sum_J (2J+1)^{1/2} (\mathbf{A}^{(e)}_{J\lambda} + \lambda \mathbf{A}^{(m)}_{J\lambda}) \tag{9.4}$$

The terms of this expansion transform under rotation as an irreducible tensor of rank J, and so, if we consider a coordinate system where $\mathbf{k}$ is described by the angles θ and φ, rather than being along the z axis

$$\hat{\xi}_{\mathbf{k}\lambda} e^{i\mathbf{k}\cdot\mathbf{r}} = -(2\pi)^{1/2} \sum_{JM} (2J+1)^{1/2} (\mathbf{A}^{(e)}_{JM} + \lambda \mathbf{A}^{(m)}_{JM}) D^{J*}_{M\lambda}(\varphi, \theta, 0) = \mathbf{A}_\lambda(\mathbf{r}) \tag{9.5}$$

Here $(\varphi, \theta, 0)$ is, as in Section 8, the rotation taking the z axis to the direction $\mathbf{k}$.

It might be shown (Blatt and Weisskopf, 1952) that, in the long-wavelength approximation and taking into account the parity selection rules, at most two values of J contribute to non-negligible effects.

The matrix element for photon emission is

$$\int \langle f|\mathbf{J}(\mathbf{r})|i\rangle \cdot \mathbf{A}^*_\lambda(\mathbf{r})\, d^3\mathbf{r} \tag{9.6}$$

According to the ideas of Section 7 we could, similarly, have obtained the matrix element for photon absorption

$$\int \langle f|\mathbf{J}(\mathbf{r})|i\rangle \cdot \mathbf{A}_\lambda(\mathbf{r})\, d^3\mathbf{r} \tag{9.7}$$

These expressions suggest a distinction between the absorption multipole operator

$$H_{JM}^{\text{abs}} = \int \mathbf{J}(\mathbf{r}) \cdot \mathbf{A}_\lambda(\mathbf{r})\, d^3\mathbf{r} \tag{9.8}$$

which transfers an angular momentum J, M to the state on which it acts and its Hermitian conjugate $(H_{JM}^{\text{abs}})^\dagger$, which removes the angular momentum J, M from the state on which it acts. The emission multipole operator is

$$H_{JM}^{\text{em}} = (H_{JM}^{\text{abs}})^\dagger = \int \mathbf{J}(\mathbf{r}) \cdot \mathbf{A}_\lambda^*(\mathbf{r})\, d^3\mathbf{r} \tag{9.9}$$

We will be concerned next with the calculation of the matrix elements (de Forest and Walecka, 1966) that are involved in (9.6) with $\mathbf{A}_\lambda(\mathbf{r})$ given by (9.5) and $\mathbf{J}(\mathbf{r})$ given by (6.5).

With $\mathbf{L} = -i\hbar\mathbf{r} \times \boldsymbol{\nabla}$ we find

$$\begin{aligned}\int \langle f|\mathbf{J}(\mathbf{r})|i\rangle \cdot \mathbf{A}_{JM}^{(e)*}\, d^3\mathbf{r} &= \frac{i^J}{k[J(J+1)]^{1/2}} \left\{ \int \langle f|\mathbf{j}(\mathbf{r})|i\rangle \cdot [\boldsymbol{\nabla} \times \mathbf{L} j_J(kr) Y_{JM}^*(\theta, \varphi)]\, d^3\mathbf{r} \right. \\ &\left. - i\hbar k^2 c \int \langle f|\boldsymbol{\mu}(\mathbf{r})|i\rangle \cdot [\mathbf{r} \times \boldsymbol{\nabla} j_J(kr) Y_{JM}^*(\theta, \varphi)]\, d^3\mathbf{r} \right\}\end{aligned} \tag{9.10}$$

Carrying out the calculations we have used the following results:

1. $\boldsymbol{\nabla} \cdot \mathbf{a} \times \mathbf{b} = \mathbf{b} \cdot \boldsymbol{\nabla} \times \mathbf{a} - \mathbf{a} \cdot \boldsymbol{\nabla} \times \mathbf{b}$; and using the fact that since the current density operator only makes sense within the boundaries of the nucleus, the integral of the divergence term extended over all space equals a surface integral over an infinite sphere which is zero.
2. $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{L}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{L}) - \boldsymbol{\nabla}^2 \mathbf{L}$ with $\boldsymbol{\nabla} \cdot \mathbf{L} = 0$ and $[\boldsymbol{\nabla}^2, \mathbf{L}] = 0$.
3. $\boldsymbol{\nabla}^2 j_J(kr) Y_{JM}^*(\theta, \varphi) = -k^2 j_J(kr) Y_{JM}^*(\theta, \varphi)$ because $\phi_{JM} = j_J(kr) Y_{JM}^*(\theta, \varphi)$ satisfies the scalar wave equation $(\boldsymbol{\nabla}^2 + k^2)\phi_{JM} = 0$.

In most cases of interest the wavelength of the nuclear radiation is large compared to the nuclear radius, kR being of the order $6 \times 10^{-3} A^{1/3} E_\gamma$ (MeV).

On account of this feature only distances such that $kr \ll 1$ will be important in (9.10), and we may keep only the first term in the expansion

$$j_J(kr) = \frac{(kr)^J}{(2J+1)!!} \left[1 - \frac{1}{2} \frac{(kr)^2}{2J+3} + \cdots \right] \tag{9.11}$$

Making use of

$$\boldsymbol{\nabla} \times \mathbf{L}\phi_{JM} = i\hbar \left\{ \boldsymbol{\nabla} \left(1 + r \frac{\partial}{\partial r} \right) - \mathbf{r}\boldsymbol{\nabla}^2 \right\} \phi_{JM}$$

we may write in the long-wavelength limit ($kr \ll 1$)

$$\boldsymbol{\nabla} \times \mathbf{L} j_J(kr) Y_{JM}^*(\theta, \varphi) \approx i\hbar k^J \frac{J+1}{(2J+1)!!} \boldsymbol{\nabla} \{ r^J Y_{JM}^*(\theta, \varphi) \} \tag{9.12}$$

Using the equality

$$\nabla\cdot(\mathbf{a}\varphi) = \varphi\nabla\cdot\mathbf{a} + \mathbf{a}\cdot\nabla\varphi \tag{9.13}$$

the first term in the curly brackets on the right of (9.10) leads to

$$-i\hbar k^J \frac{J+1}{(2J+1)!!}\int r^J Y^*_{JM}(\theta,\varphi)\nabla\cdot\langle f|\mathbf{j}(\mathbf{r})|i\rangle\, d^3\mathbf{r}$$

The continuity equation (7.16) enables us finally to write for (9.10)

$$\int \langle f|\mathbf{J}(\mathbf{r})|i\rangle\cdot\mathbf{A}^{(e)*}_{JM}\, d^3\mathbf{r}$$

$$= -i^J \frac{k^{J-1}}{(2J+1)!!}\left(\frac{J+1}{J}\right)^{1/2}\left\{\int r^J Y^*_{JM}(\theta,\varphi)\langle f|[H,\rho(\mathbf{r})]|i\rangle\, d^3\mathbf{r}\right.$$

$$\left. + \frac{i\hbar k^2 c}{J+1}\int \langle f|\boldsymbol{\mu}(\mathbf{r})|i\rangle\cdot[\mathbf{r}\times\nabla r^J Y^*_{JM}(\theta,\varphi)]\, d^3\mathbf{r}\right\} \tag{9.14}$$

We shall next compare the relative order of magnitude of the two terms in (9.14). From the operator equation

$$\nabla = \frac{\mathbf{r}}{r}\frac{\partial}{\partial r} - \frac{i}{\hbar}\frac{1}{r}\left(\frac{\mathbf{r}}{r}\times\mathbf{L}\right) \tag{9.15}$$

we may say that ∇ reduces by 1 the powers of r, and to make estimates of orders of magnitude we may replace ∇ by $1/r$. With $\mathbf{p} = -i\hbar\nabla$ and remembering the forms of $\mathbf{j}(\mathbf{r})$, (5.10), and $\boldsymbol{\mu}(\mathbf{r})$, (6.4), we may consider that $r\langle f|\mathbf{j}(\mathbf{r})|i\rangle$ and $c\langle f|\boldsymbol{\mu}(\mathbf{r})|i\rangle$ are of the same order of magnitude. Comparing, in the long-wavelength approximation, the two terms inside the curly brackets in (9.10), accounting for (9.12), we conclude that, in a rough estimate, the second term is a factor $(kr)^2$ smaller than the first. So, it is a good approximation to retain only the first term

$$\int \langle f|\mathbf{J}(\mathbf{r})|i\rangle\cdot\mathbf{A}^{(e)*}_{JM}\, d^3\mathbf{r}$$

$$= -i^J \frac{k^{J-1}}{(2J+1)!!}\left(\frac{J+1}{J}\right)^{1/2}\int r^J Y^*_{JM}(\theta,\varphi)\langle f|[H,\rho(\mathbf{r})]|i\rangle\, d^3\mathbf{r} \tag{9.16}$$

Analogously to (9.10) we may write

$$\int \langle f|\mathbf{J}(\mathbf{r})|i\rangle\cdot\mathbf{A}^{(m)*}_{JM}\, d^3\mathbf{r} = \frac{i^J}{[J(J+1)]^{1/2}}\left\{\int \langle f|\mathbf{j}(\mathbf{r})|i\rangle\cdot[\mathbf{L}j_J(kr)\,Y^*_{JM}(\theta,\varphi)]\, d^3\mathbf{r}\right.$$

$$\left. + c\int \langle f|\boldsymbol{\mu}(\mathbf{r})|i\rangle\cdot[\nabla\times\mathbf{L}j_J(kr)\,Y^*_{JM}(\theta,\varphi)]\, d^3\mathbf{r}\right\} \tag{9.17}$$

Using the formula

$$\langle f|\mathbf{j}(\mathbf{r})|i\rangle\cdot[\mathbf{r}\times\nabla j_J(kr)\,Y^*_{JM}(\theta,\varphi)] = -\mathbf{r}\times\langle f|\mathbf{j}(\mathbf{r})|i\rangle\cdot[\nabla j_J(kr)\,Y^*_{JM}(\theta,\varphi)]$$

and the result (9.12), we obtain in the long-wavelength limit

$$\int\langle f|\mathbf{J}(\mathbf{r})|i\rangle\cdot\mathbf{A}^{(m)*}_{JM}\,d^3\mathbf{r} = -\frac{i^{J-1}\hbar ck^J}{(2J+1)!!}\left(\frac{J+1}{J}\right)^{1/2}\int\left\{\frac{1}{c(J+1)}\mathbf{r}\times\langle f|\mathbf{j}(\mathbf{r})|i\rangle + \langle f|\boldsymbol{\mu}(\mathbf{r})|i\rangle\right\}\cdot[\nabla r^J Y^*_{JM}(\theta,\varphi)]\,d^3\mathbf{r} \tag{9.18}$$

We should distinguish the integrations on the nucleon coordinates and on the field coordinates.

From (9.2) and (9.3) it follows that the electric multipole operator

$$T^{(e)}_{JM} = \int\mathbf{J}(\mathbf{r})\cdot\mathbf{A}^{(e)*}_{JM}\,d^3\mathbf{r}$$

and the magnetic multipole operator

$$T^{(m)}_{JM} = \int\mathbf{J}(\mathbf{r})\cdot\mathbf{A}^{(m)*}_{JM}\,d^3\mathbf{r}$$

have opposite parity, respectively, $(-)^J$ and $(-)^{J+1}$.

Sometimes the *emission* electric multipole operator is written in the "unsafe" form

$$Q^{(e)}_{JM} = \int\rho(\mathbf{r})r^J Y^*_{JM}(\theta,\varphi)\,d^3\mathbf{r} \tag{9.19}$$

which may introduce wrong phases. Only if there is real photon emission in the transition from $|i\rangle$ to $|f\rangle(E_i > E_f)$ and because then $(E_f - E_i)/\hbar\omega = -1$, the operator

$$T^{(e)}_{JM} = -i^J\frac{k^{J-1}}{(2J+1)!!}\left(\frac{J+1}{J}\right)^{1/2}\int r^J Y^*_{JM}(\theta,\varphi)[H,\rho(\mathbf{r})]\,d^3\mathbf{r} \tag{9.20}$$

as referred in (9.16), may be written as equivalent to

$$i^J\frac{\hbar ck^J}{(2J+1)!!}\left(\frac{J+1}{J}\right)^{1/2}Q^{(e)}_{JM} \tag{9.21}$$

in the calculation of matrix elements with the final states on the left. Rose and Brink use an absorption operator, and so they write the matrix elements for emission processes with the initial state on the left (Rose and Brink, 1967). As we shall examine in Section 10, with a convenient phase choice the matrix elements are real.

From (9.18) and with

$$M_{JM}^{(m)} = \int \left\{ \frac{1}{c(J+1)} \mathbf{r} \times \mathbf{j}(\mathbf{r}) + \boldsymbol{\mu}(\mathbf{r}) \right\} \cdot \nabla r^J Y_{JM}^*(\theta, \varphi)\, d^3\mathbf{r} \tag{9.22}$$

we may write

$$T_{JM}^{(m)} = -\frac{i^{J-1}\hbar c k^J}{(2J+1)!!} \left(\frac{J+1}{J} \right)^{1/2} M_{JM}^{(m)} \tag{9.23}$$

Using (5.10) and (6.4) we may write for a system of nucleons described as points charges with magnetic moment

$$M_{JM}^{(m)} = \sum_k \frac{e}{2mc} \left[\left(g_s(k)\mathbf{s}_k + \frac{2g_l(k)}{J+1} \mathbf{l}_k \right) \cdot \nabla_k (r_k^J Y_{JM}^*(\theta_k, \varphi_k) \right] \tag{9.24}$$

From (9.19) and using (5.9) we could equally write

$$Q_{JM}^{(e)} = \sum_k g_l(k) e r_k^J Y_{JM}^*(\theta_k, \varphi_k) \tag{9.25}$$

10. THE BEHAVIOR OF THE MULTIPOLE OPERATORS UNDER HERMITIAN CONJUGATION AND TIME REVERSAL: REALITY OF THE MATRIX ELEMENTS

An irreducible tensor operator T, of rank λ, may be defined in terms of the transformation of its components under rotation of the coordinate system. Using for the D matrices the convention of Bohr and Mottelson (1969)

$$T'_{\lambda\mu} = RT_{\lambda\mu}R^{-1} = \sum_{\mu'} D^{\lambda*}_{\mu'\mu}(\alpha, \beta, \gamma) T_{\lambda\mu'} \tag{10.1}$$

Where T' is the tensor in the rotated system and $R(\alpha, \beta, \gamma)$ is the rotation operator in terms of the Euler angles α, β, and γ.

Taking the Hermitian conjugate we obtain

$$R(T_{\lambda\mu})^\dagger R^{-1} = \sum_{\mu'} D^{\lambda}_{\mu'\mu}(\alpha, \beta, \gamma)(T_{\lambda\mu'})^\dagger = \sum_{\mu'} (-)^{\mu'-\mu} D^{\lambda*}_{-\mu'-\mu}(\alpha, \beta, \gamma)(T_{\lambda\mu'})^\dagger \tag{10.2}$$

and we can still write with p arbitrary (integer with λ integer)

$$R(-)^{p+\mu}(T_{\lambda-\mu})^\dagger R^{-1} = \sum_{\mu'} D^{\lambda*}_{\mu'\mu}(\alpha, \beta, \gamma)(-)^{p+\mu'}(T_{\lambda-\mu'})^\dagger \tag{10.3}$$

That is, the components $(-)^{p+\mu}(T_{\lambda-\mu})^\dagger$ transform under rotation of the coordinate system in the same way as the operator $T_{\lambda\mu}$ and so having the correct rotational properties of a tensor operator.

Demanding $p = \lambda + 1$, we shall define the components of the Hermitian conjugate $T^\dagger$ of the tensor operator T by

$$(T^\dagger)_{\lambda\mu} = (-)^{\lambda+\mu+1}(T_{\lambda-\mu})^\dagger \tag{10.4}$$

If $T_{\lambda\mu} = (T^\dagger)_{\lambda\mu}$ the operator is Hermitian. So, we may define a Hermitian tensor operator as one whose components satisfy

$$(T_{\lambda\mu})^\dagger = (-)^{\lambda+\mu+1}T_{\lambda-\mu} \tag{10.5}$$

According to (9.2) and (9.3) the multipole operators $T^{(e)}_{JM} = \int \mathbf{J}(\mathbf{r})\cdot\mathbf{A}^{(e)*}_{JM}\, d^3\mathbf{r}$ and $T^{(m)}_{JM} = \int \mathbf{J}(\mathbf{r})\cdot\mathbf{A}^{(m)*}_{JM}\, d^3\mathbf{r}$ satisfy the condition (10.5). On the other hand the operator $Q^{(e)}_{JM}$ as expressed in (9.19) satisfies

$$(Q^{(e)}_{JM})^\dagger = (-)^M Q^{(e)}_{J-M}$$

With the definition above, as the tensor product of two tensor operators is formally identical to the coupling of two angular momentum eigenvectors, the product of two Hermitian operators is anti-Hermitian. In fact from

$$\mathfrak{T}_{\lambda\mu} = \sum_{\mu_1(\mu_2)} \langle\lambda_1\mu_1\lambda_2\mu_2|\lambda\mu\rangle T_{\lambda_1\mu_1}T_{\lambda_2\mu_2}$$

we obtain

$$\mathfrak{T}^\dagger_{\lambda\mu} = (-)^{\lambda+\mu}\mathfrak{T}_{\lambda-\mu}$$

All antiunitary operators, like the time-reversal operator $\mathscr{T}$, can be written as the product UK of a unitary operator U, which does not act on positional coordinates, by the complex conjugate operator K (Wigner, 1959). To find the explicit form for $\mathscr{T}$ we use the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi$$

It is easily proved that if there is a unitary transformation U such that $UH^*U^{-1} = H$, time-reversal invariance holds and the time-reversal state of ψ is $UK\psi$. The effect of K is tied to the representation used. So the form of the time-reversal operator depends on the nature of the Hamiltonian of the system under consideration and on the representation in which the wave function is considered. For example, if we consider the interaction Hamiltonian (6.6) depending on terms of the form $\mathbf{p}\cdot\mathbf{A}$ and $\boldsymbol{\sigma}\cdot\nabla\times\mathbf{A}$, obviously U has to contain the unitary operator U_σ acting only on spin coordinates in such a way that

$$U_\sigma\boldsymbol{\sigma}^*U_\sigma^{-1} = -\boldsymbol{\sigma} \tag{10.6}$$

and the unitary operator U_A acting on the potential vector $\mathbf{A}$ in such a way that

$$U_A\mathbf{A}U_A^{-1} = -\mathbf{A} \tag{10.7}$$

In this example the time-reversal operator would have the form

$$\mathcal{T} = U_A U_\sigma K \tag{10.8}$$

Let us examine now the form of the operator U_σ. For a spin-$\frac{1}{2}$ particle the transformation properties (10.6) of the Pauli matrices (σ_x, σ_y, σ_z) define U_σ apart from a phase.

We may write

$$U_\sigma = i\sigma_y \tag{10.9}$$

This choice makes U_σ real and so it commutes with K.

In defining phase conventions in the angular momentum representation, it is convenient to relate the time-reversal operator to the rotation operator $R_y(\pi)$ for an angle π about the y axis. We may rewrite the action of the operator U_σ, for the special case of a spin-$\frac{1}{2}$ state as a rotation by $-\pi$ about the y axis (thus reversing the z axis) of spin alone

$$U_\sigma = i\sigma_y = e^{i(\pi/2)\sigma_y} = e^{(i/\hbar)\pi s_y} = R_y^{-1}(\pi) \tag{10.10}$$

We may write

$$\begin{aligned} \mathcal{T}\begin{pmatrix}1\\0\end{pmatrix} &= U_\sigma\begin{pmatrix}1\\0\end{pmatrix} = -\begin{pmatrix}0\\1\end{pmatrix} \\ \mathcal{T}\begin{pmatrix}0\\1\end{pmatrix} &= U_\sigma\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \end{aligned} \tag{10.11}$$

and so we have

$$\mathcal{T} X_s^{m_s} = U_\sigma X_s^{m_s} = (-)^{s+m_s} X_s^{-m_s} \tag{10.12}$$

with $X_s^{m_s}$ the spinor wave function.

The factor $(-)^{s+m_s}$ is sometimes replaced by $(-)^{s-m_s}$ that could correspond to the use of $U_\sigma = -i\sigma_y$, instead of (10.9). Of course, the overall phase of the time-reversal operator is itself a matter of convention.

Next we shall consider the example of the transformation under time reversal of space–spin wave functions. For a many-particle system of A spin-$\frac{1}{2}$ particles we may choose a basis of single-particle states whose wave functions, for a potential with spherical symmetry, may be written

$$|jm\rangle = R_{nlj}(r) \sum_{m_l m_s} \langle lm_l sm_s | jm\rangle [i^l Y_{lm_l}(\theta, \varphi)] X_s^{m_s} \tag{10.13}$$

The definition of the spherical harmonics with a different phase, say $i^l Y_{lm}(\theta, \varphi)$, has always been assumed in this paper even when not explicitly stated.

The phases can always be chosen such that the radial wave function is real. From (10.12) and

$$\begin{aligned}\mathscr{T}[i^l Y_{lm_l}(\theta, \varphi)] &= K[i^l Y_{lm_l}(\theta, \varphi)] = (-)^{l+m_l}[i^l Y_{l-m_l}(\theta, \varphi)] \\ &= e^{(i/\hbar)\pi L_y}[i^l Y_{lm_l}(\theta, \varphi)]\end{aligned} \tag{10.14}$$

and because the Clebsch–Gordan coefficients have the property

$$\langle l - m_l s - m_s | j - m\rangle = (-)^{l+s-j}\langle lm_l sm_s | jm\rangle \tag{10.15}$$

our choice leads to

$$\mathscr{T}|jm\rangle = (-)^{j+m}|j - m\rangle \tag{10.16}$$

From the adopted phase convention one finds that

$$|JM\rangle = \sum_{m_1 m_2} \langle j_1 m_1 j_2 m_2 | JM\rangle |j_1 m_1\rangle |j_2 m_2\rangle$$

satisfies the transformation (10.16) if $|j_1 m_1\rangle$ and $|j_2 m_2\rangle$ already transform in accordance with it. So, the function $|JM\rangle$ constructed by combination of products of single-particle functions $|jm\rangle$ satisfies the same choice of phase. With

$$\mathscr{T} = \prod_{n=1}^{A} [i\sigma_y(n)]K \tag{10.17}$$

the many-particle wave function satisfies

$$\mathscr{T}|JM\rangle = (-)^{J+M}|J - M\rangle \tag{10.18}$$

We could have looked at this result as a consequence of having chosen, for a fixed value of J, the relative phase of the angular momentum states $|JM\rangle$ and $|JM'\rangle$, writing (Condon and Shortley convention)

$$J_\pm |JM\rangle = [J(J + 1) - M(M \pm 1)]^{1/2}|JM \pm 1\rangle \tag{10.19}$$

In fact, according to

$$\mathscr{T} J_\pm \mathscr{T}^{-1} = -J_\mp \tag{10.20}$$

we may write

$$\begin{aligned}J_\pm \mathscr{T}|JM\rangle &= -\mathscr{T} J_\mp |JM\rangle \\ &= -[J(J + 1) - M(M \mp 1)]^{1/2}\mathscr{T}|JM \mp 1\rangle\end{aligned} \tag{10.21}$$

and with the transformation (10.18) the result (10.19) would be satisfied.

The phase convention in (10.18) makes the phase real for both integral or half-integral values of J.

Since

$$R_y^{-1}(\pi)|JM\rangle = e^{(i/\hbar)\pi J_y}|JM\rangle = (-)^{J+M}|J - M\rangle \tag{10.22}$$

and with the phase conventions we have assumed, the time-reversed wave function is identical to the wave function obtained by rotation through the angle $-\pi$ about the y axis. So, by choosing the phases suitably we arrive at the conclusion that the state $|JM\rangle$ is invariant under the combined operation $R_y(\pi)\mathscr{T}$

$$R_y(\pi)\mathscr{T}|JM\rangle = |JM\rangle \tag{10.23}$$

This choice corresponds to the so-called $R_y\mathscr{T}$ convention (Alder and Steffen, 1975).

From (6.5) and using

$$\mathscr{T}\rho\mathscr{T}^{-1} = \rho, \qquad \mathscr{T}\mathbf{j}\mathscr{T}^{-1} = -\mathbf{j}, \qquad \mathscr{T}\boldsymbol{\sigma}\mathscr{T}^{-1} = -\boldsymbol{\sigma} \tag{10.24}$$

one finds

$$\mathscr{T}T^{(\pi)}_{JM}\mathscr{T}^{-1} = (-)^{J+M}T^{(\pi)}_{J-M} \tag{10.25}$$

where $\pi = e$ means electric and $\pi = m$ means magnetic.

In fact, when we wrote $\mathbf{A}$ in terms of $\mathbf{A}^{(e)}_{JM}$ and $\mathbf{A}^{(m)}_{JM}$ we were not concerned with the relative choice of phases but with a convenient right expansion. The requirement that the vector potential is odd under time reversal

$$\mathscr{T}\mathbf{A}(\mathbf{r}, t)\mathscr{T}^{-1} = -\mathbf{A}(\mathbf{r}, -t) \tag{10.26}$$

makes, in particular, Maxwell's equations and also the electromagnetic interaction, $H_{\text{int}} = -(1/c)\int \mathbf{J}\cdot\mathbf{A}\, d^3\mathbf{r}$, invariant under time reversal.

Using the Wigner–Eckart theorem the matrix element of the tensor operator may be written in terms of the reduced matrix element with the Clebsch–Gordan coefficient carrying the dependence on the orientation of coordinate frame. We obtain the result

$$\begin{aligned}
&\left\langle J_2M_2\middle|\int \mathbf{J}\cdot\mathbf{A}^{(\pi)*}_{JM}\, d^3\mathbf{r}\middle|J_1M_1\right\rangle \\
&\quad = \left\langle J_2M_2\middle|\left(\int \mathbf{J}\cdot\mathbf{A}^{(\pi)}_{JM}\, d^3\mathbf{r}\right)^{\dagger}\middle|J_1M_1\right\rangle \\
&\quad = (-)^{J+M+1}\left\langle J_2M_2\middle|\int \mathbf{J}\cdot\mathbf{A}^{(\pi)}_{J-M}\, d^3\mathbf{r}\middle|J_1M_1\right\rangle \\
&\quad = (-)^{J+M+1}\frac{\langle J_1M_1J - M|J_2M_2\rangle}{(2J_2+1)^{1/2}}\left\langle J_2\middle\|\int \mathbf{J}\cdot\mathbf{A}^{(\pi)}_{J}\, d^3\mathbf{r}\middle\|J_1\right\rangle
\end{aligned} \tag{10.27}$$

The reduced emission matrix element $\langle J_2\|\int \mathbf{J}\cdot\mathbf{A}^{(\pi)}_J\, d^3\mathbf{r}\|J_1\rangle$, which we shall represent in what follows as $\langle J_2\|T^{(\pi)}_J\|J_1\rangle$, is understood to be evaluated from that expression.

From

$$\langle J_2M_2|T^{(\pi)}_{JM}|J_1M_1\rangle = \langle J_1M_1|(T^{(\pi)}_{JM})^{\dagger}|J_2M_2\rangle^* \tag{10.28}$$

it follows that

$$\langle J_2\|T^{(\pi)}_J\|J_1\rangle = (-)^{J_1-J_2+J+1}\langle J_1\|T^{(\pi)}_J\|J_2\rangle^* \tag{10.29}$$

If the phase conventions are chosen such that the nuclear wave functions satisfy (10.18) we may write

$$\begin{aligned}\langle J_2M_2|T^{(\pi)}_{JM}|J_1M_1\rangle &= \langle J_2M_2|\mathscr{T}^{-1}\mathscr{T}T^{(\pi)}_{JM}\mathscr{T}^{-1}\mathscr{T}|J_1M_1\rangle \\ &= (-)^{J_2+M_2+J_1+M_1+J+M}\langle J_2 - M_2|T^{(\pi)}_{J-M}|J_1 - M_1\rangle^* \end{aligned} \tag{10.30}$$

Applying the Wigner–Eckart theorem to the right-hand side of (10.30) and using symmetry properties of the Clebsch–Gordan coefficients it follows that, with the phase conventions given above, the reduced matrix elements are real:

$$\langle J_2\|T^{(\pi)}_J\|J_1\rangle = \langle J_2\|T^{(\pi)}_J\|J_1\rangle^* \tag{10.31}$$

and so using (10.29)

$$\langle J_2\|T^{(\pi)}_J\|J_1\rangle = (-)^{J_1-J_2+J+1}\langle J_1\|T^{(\pi)}_J\|J_2\rangle \tag{10.32}$$

From (9.21) and (9.23) we might write (Steffen, 1971) the relation between the reduced matrix elements as defined above and the reduced matrix elements relative to the operators $\mathfrak{M}^{(\pi)}_{JM}$ as considered by Bohr and Mottelson (1969)

$$\langle J_2\|T^{(e)}_J\|J_1\rangle = i^J\frac{\hbar ck^J}{(2J+1)!!}\left(\frac{J+1}{J}\right)^{1/2}\langle J_2\|\mathfrak{M}^{(e)}_J\|J_1\rangle$$

$$\langle J_2\|T^{(m)}_J\|J_1\rangle = -i^{J-1}\frac{\hbar ck^J}{(2J+1)!!}\left(\frac{J+1}{J}\right)^{1/2}\langle J_2\|\mathfrak{M}^{(m)}_J\|J_1\rangle$$

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